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A heterogeneous system with finite waiting space

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SUMMARY

A simple queue with two heterogeneous servers is analyzed. The emphasis is on comparing the two-server heterogeneous and homogeneous systems with the restriction of an upper limit N on the queue size. The optimal service rates for both the servers are found in terms of the arrival rate and the traffic intensity ρ. The average characteristics of the heterogeneous system are minimized, and their improvement over the corresponding homogeneous system characteristics is established. For different values of N and ρ, tables are given which compare the average characteristics of the two systems.

1. Introduction

In a queuing system with more than one server, one is confronted with the problem of whether the servers should have equal service rates or different service rates. The former situation is referred to as a queuing system with homogeneous servers, the later as a queuing system with heterogeneous servers. Unless the situation is mechanically controlled, the case of heterogeneous servers is more applicable in practice, but it is hard to study from a mathematical point of view.

In this paper we analyze a Markovian queuing system with heterogeneous servers under the assumption of finite waiting space. We investigate the condition under which a two server heterogeneous system is better than the corresponding homogeneous system. This condition only involves the traffic intensity, ρ, and the service rates. The two systems are compared by replacing the service rate for each server in the homogeneous system by the average service rate of the corresponding heterogeneous system.

The heterogeneous system under consideration is such that one server is faster than the other and there cannot be more than N customers in the system at any time. The expressions for the optimal service rates of the two servers are obtained in terms of the arrival rate and the traffic intensity, ρ. These expressions are such that they minimize the average characteristics of the heterogeneous system and yield an improvement over the corresponding homogeneous system. For different values of N and ρ, Table 1 compares the average number of the customers and the actual queue size in both the homogeneous and the heterogeneous systems.

The related work on this problem has been discussed by Singh [1, 2].

2. Statement of the problem

Consider a queuing system with two servers, which is completely specified by the following:
Arrival Pattern:
Customers arrive to join a common waiting line, following the Poisson Law with mean arrival rate, λ.
Service Mechanism:
The service times of the two servers are independently and negative exponentially distributed with mean service rates μ_1 and μ_2 ($\mu_1 > \mu_2$) respectively.
Queue Discipline:
A customer arrives to find:

1. Both servers free; he chooses the first server, since here he expects to get the faster service.
2. Both servers busy; he joins the system as long as there are less than N customers in the system and waits in a line in order of arrival. The customer at the top of the queue occupies the server which becomes available first. After joining the system a customer does not renege.
3. Only one server busy; he chooses the free server.

The problem is to study this process ($M/M_i/2/N$) and compare it with the corresponding homogeneous system ($M/M/2/N$).

3. Analysis of the $M/M_i/2/N$ system

Let

$P_n(t)$ = probability that there are n customers in the system at time, t.

For

$$n = 1, \quad P_1(t) = P_{10}(t) + P_{01}(t), \tag{i}$$

where

$P_{10}(t)$ = probability that the first server is occupied, and

$P_{01}(t)$ = probability that the second server is occupied.

Following the usual arguments, the differential-difference equations for the $M/M_i/2/N$ system can be written as follows:

$$P'_0(t) = \mu_1 P_{10}(t) + \mu_2 P_{01}(t) - \lambda P_0(t).$$

$$P'_{10}(t) = -(\lambda + \mu_1)P_{10}(t) + \mu_2 P_2(t) + \lambda P_0(t).$$

$$P'_{01}(t) = -(\lambda + \mu_2)P_{01}(t) + \mu_1 P_2(t).$$

$$P'_n(t) = -(\lambda + \mu)P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}(t), \quad 1 < n < N.$$

$$P'_N(t) = -\mu P_N(t) + \lambda P_{N-1}(t),$$

where

$\mu = \mu_1 + \mu_2$ and $P'_n(t) = dP_n(t)/dt$.

The following steady-state solution for the above system of equations can be easily verified:

$$P_0 = \frac{1}{1+C}, \tag{2}$$

where

$$C = \frac{1-\rho^N}{1-\rho} \, \frac{\lambda(\lambda+\mu_2)}{\mu_1\mu_2(1+2\rho)} \tag{3}$$

$$\rho = \frac{\lambda}{\mu_1+\mu_2}.$$

$$P_1 = \frac{\lambda(\lambda+\mu_2)}{\mu_1\mu_2(1+2\rho)} P_0. \tag{4}$$

$$P_n = \rho^{n-1}P_1, \quad 2 \leqq n \leqq N. \tag{5}$$

The average characteristics of the system are:

(i) Average number of customers in the system:

$$E(Q) = \sum_{n=0}^{N} nP_n = [1-(N+1)\rho^N + N\rho^{N+1}] \frac{P_1}{(1-\rho)^2}. \tag{6}$$

(ii) Average queue length:

$$E(Q') = \sum_{n=3}^{N} (n-2)P_n = [1-(N-1)\rho^{N-2} + (N-2)\rho^{N-1}]\left(\frac{\rho}{1-\rho}\right)^2 P_1. \tag{7}$$

Similarly for the homogeneous system $M/M/2/N$, it can be verified that:

$$P_0 = \frac{1-\rho}{1+\rho-2\rho^{N+1}}, \tag{8}$$

$$P_1 = 2\rho P_0, \tag{9}$$

and

$$P_n = \rho^{n-1}P_1, \quad 2 \leqq n \leqq N.$$

The average characteristics in terms of P_1 are the same as that of the $M/M_i/2/N$ system.

4. Comparison between $M/M_i/2/N$ and $M/M/2/N$ systems

In order to distinguish the state probabilities and the average characteristics of the heterogeneous system, we use the asterisks on the symbols. For example, P_0^* denotes the probability that there are no customers in the heterogeneous system. For comparison purposes we replace the service rate, μ, of the homogeneous system by the average service rate, $\mu_1 + \mu_2/2$, of the corresponding heterogeneous system. It is to be pointed out that this substitution leaves ρ unaffected for both the systems. Furthermore the difference between the two systems lies in the tail with probabilities for $n < 2$, otherwise both have the geometric distribution for $n \geqq 2$. As in Singh [2] we define a heterogeneous system to be "better" than the corresponding homogeneous system if $P_0^* > P_0$ and $P_n^* < P_n$ for $n \geqq 1$.

The following theorem then gives a condition in terms of ρ and the service rates guaranteeing that the $M/M_i/2/N$ system is better than the corresponding $M/M/2/N$ system. It is to be noted that this condition is independent of N.

Theorem I: *Given ρ and λ, a necessary and sufficient condition that a two server heterogeneous system is better than the corresponding homogeneous system under the assumption of the finite waiting space is $\rho < (\mu_2/(\mu_1 - \mu_2))$. Then,*

$$\delta E(\cdot) = E(\cdot) - E^*(\cdot) \geqq 0.$$

Proof: Recall that for both the systems $P_n = \rho^{n-1} P_1$ for $n \geqq 2$, hence it suffices to compare the state probabilities for $n = 0$ and 1.

From eqs. (8) and (9), we have

$$P_1 = 2\rho P_0 = \left[\frac{1}{2\rho} + \frac{1-\rho^N}{1-\rho}\right]^{-1}. \tag{10}$$

From eqs. (2), (3) and (4) we have

$$P_1^* = \frac{\lambda(\lambda + \mu_2)P_0^*}{\mu_1\mu_2(1+2\rho)} = \left[\frac{\mu_1\mu_2(1+2\rho)}{\lambda(\lambda+\mu_2)} + \frac{1-\rho^N}{1-\rho}\right]^{-1}. \tag{11}$$

Now from eqs. (10) and (11) it follows that

$$P_1^* < P_1 \Leftrightarrow \frac{\lambda(\lambda+\mu_2)}{\mu_1\mu_2(1+2\rho)} < 2\rho. \tag{12}$$

Note that the above inequality also guarantees that $P_0^* > P_0$. Algebraic simplification of eq. (12) gives

$$\rho < \frac{\mu_2}{\mu_1 - \mu_2}. \tag{13}$$

In order to show $\delta E(\ ) \geqq 0$, we use eqs. (6) and (7).

$$\delta E(Q) = E(Q) - E^*(Q) = [1 - (N+1)\rho^N + N\rho^{N+1}](P_1 - P_1^*)(1-\rho)^{-2}. \tag{14}$$

$$\delta E(Q') = E(Q') - E^*(Q') = [1 - (N-1)\rho^{N-2} + (N-2)\rho^{N-1}](P_1 - P_1^*)\rho^2(1-\rho)^{-2}. \tag{15}$$

In the two expressions above, the quantities in the brackets are non-negative and $(P_1 - P_1^*)$ is positive, from eqs. (12) and (13). Therefore it follows that $\delta E(\cdot) \geqq 0$.

Our next theorem gives the best allocation of the service rates, μ_1 and μ_2, to the two servers, minimizes $E^*(\cdot)$ and yields an improvement over the corresponding homogeneous system. In this context we first prove a lemma which will be used in the theorem.

Lemma: *For fixed λ and ρ, $f(\mu_2) = (\lambda + \mu_2)/[\mu_2(\lambda - \rho\mu_2)]$ is a strictly convex function of μ_2 and achieves a unique minimum at $\mu_2^0 = \lambda[(1+\rho^{-1})^{\frac{1}{2}} - 1]$.*

Proof: It is easy to see that

$$f(\mu_2) = \frac{1}{\mu_2} + \left(\frac{1+\rho}{\rho}\right)\frac{1}{(\lambda/\rho - \mu_2)}.$$

Let

$$f_1(\mu_2) = \frac{1}{\mu_2} \text{ and } f_2(\mu_2) = \frac{1}{(\lambda/\rho - \mu_2)}.$$

Then

$$f_1''(\mu_2) = \frac{2}{\mu_2^3} > 0 \text{ and } f_2''(\mu_2) = \frac{2}{(\lambda/\rho - \mu_2)^3} = \frac{2}{\mu_1^3} > 0,$$

and therefore $f_1(\mu_2)$ and $f_2(\mu_2)$ are strictly convex functions of μ_2. It is obvious that $f(\mu_2)$ can be expressed as a positive linear combination of $f_1(\mu_2)$ and $f_2(\mu_2)$. Now strict convexity of $f(\mu_2)$ follows from the fact that the positive linear combination of two strictly convex functions is again strictly convex.

In order to determine the minimum, we find that the stationary points are given by

$$f'(\mu_2) = -\frac{1}{\mu_2^2} + \left(\frac{1+\rho}{\rho}\right)\frac{1}{(\lambda/\rho - \mu_2)^2} = 0$$

i.e.,

$$(1+\rho)\mu_2^2 - \rho\left(\frac{\lambda}{\rho} - \mu_2\right)^2 = 0$$

i.e.,

$$\rho\mu_2^2 + 2\lambda\rho\mu_2 - \lambda^2 = 0.$$

This is a quadratic in μ_2 and has the following two roots:

$$\lambda(\sqrt{1+\rho^{-1}} - 1) \text{ and } -\lambda(\sqrt{1+\rho^{-1}} + 1)$$

since we cannot have μ_2 negative, therefore the only choice is $\mu_2^0 = \lambda(\sqrt{1+\rho^{-1}} - 1)$. The fact that this is a unique minimum follows from strict convexity of $f(\mu_2)$.

Theorem II: *For given N, λ and ρ, $\mu_2^0 = \lambda[(1+\rho^{-1})^{\frac{1}{2}} - 1]$ is the best allocation to the second (i.e., slower) server. This μ_2^0 minimizes $E^*(\cdot)$ of the $M/M_i/2/N$ system and gives the following reductions over the corresponding homogeneous system:*

$$\delta E(Q) = [1 - (N+1)\rho^N + N\rho^{N+1}]K, \tag{16}$$

$$\delta E(Q') = [1 - (N-1)\rho^{N-2} + (N-2)\rho^{N-1}]\rho^2 K, \tag{17}$$

where

$$K = \frac{1}{1-\rho}\left[\frac{2\rho}{1+\rho-2\rho^{N+1}} - \frac{1}{1+(1-\rho)(1+2\rho)\left(\sqrt{\frac{1+\rho}{\rho}} - 1\right)^2 - \rho^N}\right] > 0. \tag{18}$$

Proof: From eqs. (3) and (4), after substituting $\mu_1 = \lambda/\rho - \mu_2$ it can be seen that P_1^* is a convex function of μ_2.

$$P_1^* = \frac{1-\rho}{1-\rho^N}\left[1 - \frac{1}{1 + \dfrac{\rho(1-\rho^N)\lambda}{(1-\rho)(1+2\rho)} f(\mu_2)}\right]. \tag{19}$$

Now it follows from the Lemma that $\mu_2^0 = \lambda\{[\sqrt{(1+\rho)/\rho}] - 1\}$ minimizes P_1^*. It is easy to see that μ_2^0 maximizes P_0^* and minimizes P_n^* for $1 \leqq n \leqq N$. For fixed N, λ and ρ, $E^*(\cdot)$ only depend upon P_1^* and therefore μ_2^0 also minimizes $E^*(\cdot)$.

We next observe that

$$\lambda f(\mu_2^0) = \rho^{-1}[(1+\rho^{-1})^{\frac{1}{2}} - 1]^{-2} \tag{20}$$

and therefore from eq. (19) we have after algebraic simplification

$$P_1^* = \frac{(1-\rho)}{1 - \rho^N + (1-\rho)(1+2\rho)\left(\sqrt{\dfrac{1+\rho}{\rho}} - 1\right)^2}. \tag{21}$$

Now eqs. (16) and (17) follow from eqs. (14) and (15) and the fact that

$$P_1 - P_1^* = (1-\rho)^2 K$$

where

$$K = \frac{1}{1-\rho}\left[\frac{2\rho}{1+\rho-2\rho^{N+1}} - \frac{1}{1 + (1-\rho)(1+2\rho)\left(\sqrt{\dfrac{1+\rho}{\rho}} - 1\right)^2 - \rho^N}\right].$$

In order to show that K is positive, we assume on the contrary that $K < 0$, then

$$2\rho + 2\rho(1-\rho)(1+2\rho)\left(\sqrt{\frac{1+\rho}{\rho}} - 1\right)^2 - 2\rho^{N+1} \leqq 1 + \rho - 2\rho^{N+1},$$

i.e., $2\rho(1+2\rho)\left(\sqrt{\dfrac{1+\rho}{\rho}} - 1\right)^2 \leqq 1,$

i.e., $2(1+2\rho)^2 - 1 \leqq 4\rho(1+2\rho)\sqrt{\dfrac{1+\rho}{\rho}},$

i.e., $4(1+2\rho)^4 - 4(1+2\rho)^2 + 1 \leqq 16\rho(1+2\rho)^2(1+\rho),$

which is impossible. Therefore K is positive.

In order to see the validity of Theorem II, for different values of N, we present in Table 1 the average number of customers $E(Q)$ and the actual queue length $E(Q')$ for the $M/M/2/N$ and $M/M_i/2/N$ systems. These tables are for $N = 5, 10, 20, 30, 40, 50, 75, 100, 125$ and 150, and $\rho = .1, .2, \ldots, .9, .999$. It is to be noted that the value $\rho = 1$ corresponds to

TABLE 1

Average characteristics in $M/M/2/N$ and $M/M_i/2/N$ system for different values of N.

Table for $N = 5$

ρ	$E(Q)$	$E^*(Q)$	$E(Q')$	$E^*(Q')$	$\%E(\cdot)$
0.1	0.2020	0.1635	0.0020	0.0016	19.0711
0.2	0.4160	0.3722	0.0162	0.0145	10.5327
0.3	0.6529	0.6122	0.0544	0.0510	6.2278
0.4	0.9187	0.8837	0.1258	0.1210	3.8153
0.5	1.2128	1.1838	0.2340	0.2285	2.3858
0.6	1.5266	1.5036	0.3762	0.3705	1.5112
0.7	1.8474	1.8295	0.5438	0.5385	0.9663
0.8	2.1612	2.1478	0.7256	0.7211	0.6232
0.9	2.4570	2.4470	0.9109	0.9072	0.4057
1.0	2.7247	2.7174	1.0892	1.0862	0.2680

Table for $N = 10$

ρ	$E(Q)$	$E^*(Q)$	$E(Q')$	$E^*(Q')$	$\%E(\cdot)$
0.1	0.2020	0.1635	0.0020	0.0016	19.0711
0.2	0.4167	0.3728	0.0167	0.0149	10.5317
0.3	0.6593	0.6183	0.0593	0.0556	6.2212
0.4	0.9517	0.9156	0.1518	0.1461	3.7940
0.5	1.3264	1.2954	0.3270	0.3194	2.3388
0.6	1.8266	1.8005	0.6302	0.6212	1.4307
0.7	2.4929	2.4716	1.1072	1.0978	0.8533
0.8	3.3307	3.3144	1.7729	1.7642	0.4889
0.9	4.2814	4.2700	2.5801	2.5732	0.2673
1.0	5.2289	5.2215	3.4202	3.4154	0.1409

Table for $N = 20$

ρ	$E(Q)$	$E^*(Q)$	$E(Q')$	$E^*(Q')$	$\%E(\cdot)$
0.1	0.2020	0.1635	0.0020	0.0016	19.0711
0.2	0.4167	0.3728	0.0167	0.0149	10.5317
0.3	0.6593	0.6183	0.0593	0.0556	6.2212
0.4	0.9524	0.9162	0.1524	0.1466	3.7938
0.5	1.3333	1.3022	0.3333	0.3255	2.3373
0.6	1.8744	1.8477	0.6745	0.6648	1.4244
0.7	2.7316	2.7088	1.3320	1.3208	0.8342
0.8	4.2316	4.2127	2.6357	2.6240	0.4470
0.9	8.8018	6.7880	5.0278	5.0176	0.2024
1.0	10.2088	10.2014	8.3073	8.3013	0.0728

TABLE 1, continued

Table for N = 30

ρ	$E(Q)$	$E^*(Q)$	$E(Q')$	$E^*(Q')$	$\%E(\cdot)$
0.1	0.2020	0.1635	0.0020	0.0016	19.0711
0.2	0.4167	0.3728	0.0167	0.0149	10.5317
0.3	0.6593	0.6183	0.0593	0.0556	6.2212
0.4	0.9524	0.9162	0.1524	0.1466	3.7938
0.5	1.3333	1.3022	0.3333	0.3255	2.3373
0.6	1.8750	1.8483	0.6750	0.6654	1.4243
0.7	2.7445	2.7217	1.3445	1.3333	0.8336
0.8	4.4108	4.3912	2.8112	2.7988	0.4430
0.9	8.1965	8.1812	6.4048	6.3929	0.1866
1.0	15.1683	15.1609	13.2348	13.2284	0.0490

Table for N = 40

ρ	$E(Q)$	$E^*(Q)$	$E(Q')$	$E^*(Q')$	$\%E(\cdot)$
0.1	0.2020	0.1635	0.0020	0.0016	19.0711
0.2	0.4167	0.3728	0.0167	0.0149	10.5317
0.3	0.6593	0.6183	0.0593	0.0556	6.2212
0.4	0.9524	0.9162	0.1524	0.1466	3.7938
0.5	1.3333	1.3022	0.3333	0.3255	2.3373
0.6	1.8750	1.8483	0.6750	0.6654	1.4243
0.7	2.7451	2.7222	1.3451	1.3339	0.8336
0.8	4.4397	4.4200	2.8397	2.8271	0.4425
0.9	8.8981	8.8820	7.1010	7.0881	0.1817
1.0	20.1101	20.1027	18.1605	18.1537	0.0371

Table for N = 50

ρ	$E(Q)$	$E^*(Q)$	$E(Q')$	$E^*(Q')$	$\%E(\cdot)$
0.1	0.2020	0.1635	0.0020	0.0016	19.0711
0.2	0.4167	0.3728	0.0167	0.0149	10.5317
0.3	0.6593	0.6183	0.0593	0.0556	6.2212
0.4	0.9524	0.9162	0.1524	0.1466	3.7938
0.5	1.3333	1.3022	0.3333	0.3255	2.3373
0.6	1.8750	1.8483	0.6750	0.6654	1.4243
0.7	2.7451	2.7222	1.3451	1.3339	0.8336
0.8	4.4438	4.4241	2.8438	2.8312	0.4425
0.9	9.2258	9.2092	7.4268	7.4134	0.1800
1.0	25.0349	25.0274	23.0755	23.0686	0.0299

Table for N = 75

ρ	$E(Q)$	$E^*(Q)$	$E(Q')$	$E^*(Q')$	$\%E(\cdot)$
0.1	0.2020	0.1635	0.0020	0.0016	19.0711
0.2	0.4167	0.3728	0.0167	0.0149	10.5317
0.3	0.6593	0.6183	0.0593	0.0556	6.2212

TABLE 1, continued

0.4	0.9524	0.9162	0.1524	0.1466	3.7938
0.5	1.3333	1.3022	0.3333	0.3255	2.3373
0.6	1.8750	1.8483	0.6750	0.6654	1.4243
0.7	2.7451	2.7222	1.3451	1.3339	0.8336
0.8	4.4444	4.4248	2.8444	2.8319	0.4425
0.9	9.4472	9.4303	7.6473	7.6336	0.1792
1.0	37.2731	37.2655	35.3006	35.2934	0.0203

Table for N = 100

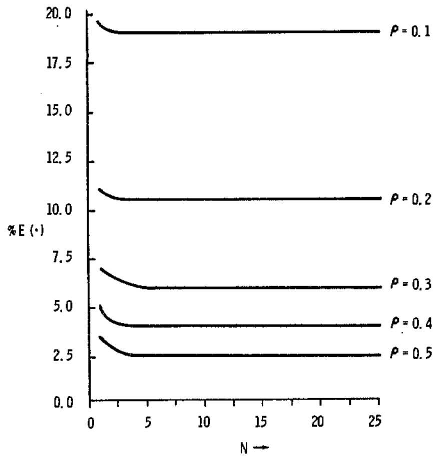
ρ	$E(Q)$	$E^*(Q)$	$E(Q')$	$E^*(Q')$	$\%E(\cdot)$
0.1	0.2020	0.1635	0.0020	0.0016	19.0711
0.2	0.4167	0.3728	0.0167	0.0149	10.5317
0.3	0.6593	0.6183	0.0593	0.0556	6.2212
0.4	0.9524	0.9162	0.1524	0.1466	3.7938
0.5	1.3333	1.3022	0.3333	0.3255	2.3373
0.6	1.8750	1.8483	0.6750	0.6654	1.4243
0.7	2.7451	2.7222	1.3451	1.3339	0.8336
0.8	4.4444	4.4248	2.8444	2.8319	0.4425
0.9	9.4712	9.4542	7.6712	7.6574	0.1792
1.0	49.4067	49.3991	47.4277	47.4203	0.0154

Table for N = 125

ρ	$E(Q)$	$E^*(Q)$	$E(Q')$	$E^*(Q')$	$\%E(\cdot)$
0.1	0.2020	0.1635	0.0020	0.0016	19.0711
0.2	0.4167	0.3728	0.0167	0.0149	10.5317
0.3	0.6593	0.6183	0.0593	0.0556	6.2212
0.4	0.9524	0.9162	0.1524	0.1466	3.7938
0.5	1.3333	1.3022	0.3333	0.3255	2.3373
0.6	1.8750	1.8483	0.6750	0.6654	1.4243
0.7	2.7451	2.7222	1.3451	1.3339	0.8336
0.8	4.4444	4.4248	2.8444	2.8319	0.4425
0.9	9.4735	9.4565	7.6735	7.6597	0.1792
1.0	61.4361	61.4284	59.4531	59.4456	0.0125

Table for N = 150

ρ	$E(Q)$	$E^*(Q)$	$E(Q')$	$E^*(Q')$	$\%E(\cdot)$
0.1	0.2020	0.1635	0.0020	0.0016	19.0711
0.2	0.4167	0.3728	0.0167	0.0149	10.5317
0.3	0.6593	0.6183	0.0593	0.0556	6.2212
0.4	0.9524	0.9162	0.1524	0.1466	3.7938
0.5	1.3333	1.3022	0.3333	0.3255	2.3373
0.6	1.8750	1.8483	0.6750	0.6654	1.4243
0.7	2.7451	2.7222	1.3451	1.3339	0.8338
0.8	4.4444	4.4248	2.8444	2.8319	0.4425
0.9	9.4737	9.4567	7.6737	7.6599	0.1791
1.0	73.3614	73.3536	71.3757	71.3681	0.0105



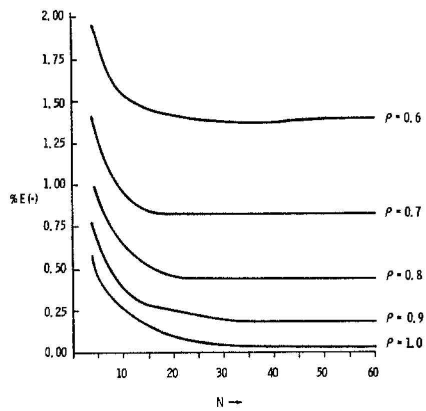


Fig. 1. Percentage reductions vs N where $\varrho = 0.1$–0.5. Fig. 2. Percentage reductions vs N where $\varrho = 0.6$–1.0.

$\rho = .999$. The last column in these tables represents the percentage reductions $\%E(\cdot)$ in the average characteristics for both the systems, and is computed as follows:

$$\%E(\cdot) = \frac{E(\cdot) - E^*(\cdot)}{E(\cdot)} \times 100 = \frac{\delta E(\cdot)}{E(\cdot)} \times 100.$$

Finally we plot these percentage reductions as functions of N in Figs. 1 and 2. In Fig. 1, different plots are for $\rho = .1, .2, .3, .4$ and $.5$. Fig. 2 contains the plots for $\rho = .6, .7, .8, .9$ and $.999$ ($\cong 1$). These tables and plots indicate that for $\rho \leqq .5$, the heterogeneous system is better than the homogeneous system whenever the upper limit on queue size is $N \leqq 10$. Whereas for $\rho > .5$, $M/M_i/2/N$ system is better than the $M/M/2/N$ system for $N \leqq 50$.

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